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Even and Odd Coherent Vectors in a Deformed Hilbert Space

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We discuss several properties of even, odd, and orthogonal-even coherent vectors in a deformed Hilbert space.

1. INTRODUCTION

We consider the set

$$H_q = \{f: f(z) = \Sigma a_n z^n \text{ where } \Sigma[n]! \left|a_n\right|^2 < \infty\}$$

where $[n] = (1 - q^n)/(1 - q)$, $0 < q < 1$.

For $f, g \in H_q$, $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$, $g(z) = \Sigma_{n=0}^{\infty} b_n z^n$, we define addition and scalar multiplication as follows:

$$(f + g)(z) = f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n \tag{1}$$

and

$$(\lambda \circ f)(z) = \lambda \circ f(z) = \sum_{n=0}^{\infty} \lambda a_n z^n \tag{2}$$

It is easily seen that H_q forms a vector space with respect to usual pointwise scalar multiplication and pointwise addition by (1) and (2). We observe that $e_q(z) = \Sigma_{n=0}^{\infty} (z^n/[n]!)$ belongs to H_q.

Now we define the inner product of two functions $f(z) = \Sigma\, a_n z^n$ and $g(z) = \Sigma\, b_n z^n$ belonging to H_q as

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$$(f, g) = \Sigma[n]!\, \bar{a}_n\, b_n \tag{3}$$

The corresponding norm is given by

$$\|f\|^2 = (f, f) = \Sigma[n]!\, |a_n|^2 < \infty$$

With this norm derived from the inner product it can be shown that H_q is a complete normed space. Hence H_q forms a Hilbert space.

In a recent paper [1] we have proved that the set $\{z^n / \sqrt{[n]!},\ n = 0, 1, 2, 3, \ldots\}$ forms a complete orthonormal set. If we consider the actions on H_q

$$Tf_n = \sqrt{[n]}\, f_{n-1}$$

$$T^* f_n = \sqrt{[n+1]}\, f_{n+1}$$

where T is the backward shift and its adjoint T^* is the forward shift operator on H_q, then we have shown [1] that the solution of the eigenvalue equation

$$Tf_\alpha = \alpha f_\alpha \tag{4}$$

is given by

$$f_\alpha = e_q(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}} f_n \tag{5}$$

We call f_α a *coherent vector* in H_q.

This paper is divided into four sections. In Section 1 we state coherent vectors in H_q. In Section 2 we discuss superposition of coherent vectors. In particular, we introduce even and odd coherent vectors in H_q. In Section 3 we discuss quadrature variance for even and odd coherent vectors and state when they exhibit squeezing. Also we study their antibunching properties. In Section 4 we study similar properties for orthogonal-even coherent vectors.

2. SUPERPOSITION OF COHERENT VECTORS

We consider a vector of the form

$$g_\alpha = A(f_\alpha + re^{i\phi} f_{-\alpha}) \tag{6}$$

where f_α and $f_{-\alpha}$ are coherent vectors (5) and r and ϕ are real parameters.

To normalize we observe that

$$\begin{aligned} 1 = (g_\alpha, g_\alpha) &= A^2(f_\alpha + re^{i\phi} f_{-\alpha}, f_\alpha + re^{i\phi} f_{-\alpha}) \\ &= A^2[(f_\alpha, f_\alpha) + r^2(f_{-\alpha}, f_{-\alpha}) \\ &\quad + re^{i\phi}(f_\alpha, f_{-\alpha}) + re^{-i\phi}(f_{-\alpha}, f_\alpha)] \\ &= A^2[1 + r^2 + re^{i\phi} e_q(|\alpha|^2)^{-1}\, e_q(-|\alpha|^2) \end{aligned}$$

$$+ re^{-i\phi} e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)]$$
$$= A^2[1 + r^2 + 2re_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)\cos\phi] \tag{7}$$

From (6) and (7) we have

$$g_\alpha = \frac{1}{\sqrt{1 + r^2 + 2re_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)\cos\phi}}(f_\alpha + re^{i\phi} f_{-\alpha}) \tag{8}$$

This superposition (8) is of special kind, as it is the eigenvector of T^2, the square of the backwardshift:

$$T^2 g_\alpha = \alpha^2 g_\alpha \tag{9}$$

In the case $r = 0$, the vector (8) reduces to the coherent vector (5).

2.1. Even and Odd Coherent Vectors

In the case $r = 1$, $\phi = 0$ the vector (8) reduces to

$$f_\alpha^e = \frac{f_\alpha + f_{-\alpha}}{\sqrt{2(1 + e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)}} \tag{10}$$

In the case $r = 1$, $\phi = \pi$ the vector (8) reduces to

$$f_\alpha^o = \frac{f_\alpha - f_{-\alpha}}{\sqrt{2(1 - e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)}} \tag{11}$$

We call f_α^e an *even coherent vector* and f_α^o an *odd coherent vector*. Here we shall study some properties of even and odd coherent vectors.

We observe that

$$\begin{aligned}
(f_\alpha^e, f_\alpha^o) &= \frac{1}{2\sqrt{1 - e_q(|\alpha|^2)^{-2} e_q(-|\alpha|^2)^2}}(f_\alpha + f_{-\alpha}, f_\alpha - f_{-\alpha}) \\
&= \frac{1}{2\sqrt{1 - e_q(|\alpha|^2)^{-2} e_q(-|\alpha|^2)^2}}[(f_\alpha, f_\alpha) - (f_\alpha, f_{-\alpha}) \\
&\quad + (f_{-\alpha}, f_\alpha) - (f_{-\alpha}, f_{-\alpha})] \\
&= \frac{1}{2\sqrt{1 - e_q(|\alpha|^2)^{-2} e_q(-|\alpha|^2)^2}}[1 - e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2) \\
&\quad + e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2) - 1] \\
&= 0
\end{aligned} \tag{12}$$

Again,

$$Tf_\alpha^e = \frac{1}{\sqrt{2(1 + e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)}}[Tf_\alpha + Tf_{-\alpha}]$$
$$= \frac{1}{\sqrt{2(1 + e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)}}[\alpha f_\alpha - \alpha f_{-\alpha}]$$
$$= \frac{\sqrt{2(1 - e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)}}{\sqrt{2(1 + e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)}} \alpha f_\alpha^o \tag{13}$$

Similarly,

$$Tf_\alpha^o = \frac{\sqrt{2(1 + e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)}}{\sqrt{2(1 - e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)}} \alpha f_\alpha^e \tag{14}$$

Thus, arbitrary even and odd coherent vectors are orthogonal and can be exchanged by the operator T.

3. QUADRATURE VARIANCE

We may write the square of the backwardshift T^2 as

$$T^2 = T_1 + iT_2 \tag{15}$$

where

$$T_1 = \frac{1}{2}(T^2 + T^{*2}), \qquad T_2 = \frac{1}{2i}(T^2 - T^{*2}) \tag{16}$$

For arbitrary operator T, the *variance* $(\Delta T)^2$ of T is defined by

$$(\Delta T)^2 = \langle T^2 \rangle - \langle T \rangle^2 \tag{17}$$

where $\langle T \rangle = (\phi, T\phi)$ for arbitrary vector ϕ in Hilbert space.

Now we observe that

$$TT^* = [N + 1], \qquad T^*T = [N]$$

where the operator N is such that

$$Nf_n = nf_n$$

Also we can verify that

$$NT - TN = -T, \qquad NT^* - T^*N = T^*$$

and

$$TT^* - T^*T = q^N$$

We can also show that q^N commutes with both T^*T and TT^*.

By (16) we observe that $T_1T_2 - T_2T_1 = iT_o$, where

$$T_o = \frac{1}{2}(q^{n-1} + q^n)(q + 1)T^*T + \frac{1}{2}q^{2n}(q + 1) \tag{18}$$

In calculating (18), we have utilized

$$TT^* - T^*T = q^n \tag{19}$$

$$T^2T^{*2} - T^{*2}T^2 = (q^{n-1} + q^n)(q + 1)T^*T + q^{2n}(q + 1)$$

Now, when the expectation values $\langle T_1^2\rangle$, $\langle T_1\rangle^2$, $\langle T_2^2\rangle$, and $\langle T_2\rangle^2$ are calculated for the superposition $g_\alpha \in H_q$ as in (8), we obtain

$$\begin{aligned}(\Delta T_1)^2 = (\Delta T_2)^2 &= \frac{1}{2}\langle T_o\rangle \\ &= \frac{1}{4}(q^{n-1} + q^n)(q + 1)|\alpha|^2 \\ &\times \frac{1 + r^2 - 2re_q(|\alpha|^2)^{-1}e_q(-|\alpha|^2)\cos\phi}{1 + r^2 + 2re_q(|\alpha|^2)^{-1}e_q(-|\alpha|^2)\cos\phi} + \frac{1}{4}q^{2n}(q + 1)\end{aligned} \tag{20}$$

Thus they furnish an equality in the *uncertainty relation*:

$$(\Delta T_1)^2(\Delta T_2)^2 \geq \frac{1}{4}\langle T_o\rangle^2 \tag{21}$$

The coherent vector with $\Delta T_1 = \Delta T_2$ is a special case of a more general class of vectors which may have reduced uncertainty in one quadrature at the expense of increased uncertainty in the other such that

$$\Delta T_1 < 1 < \Delta T_2 \tag{22}$$

These vectors are called *squeezed vectors*. Equation (20) indicates that superposition g_α cannot exhibit squeezing for the square of the backwardshift T^2.

3.1. Quadrature Variance with Even and Odd Coherent Vectors

If we specialize (8) for even and odd coherent vectors f_α^e (10) and f_α^o (11), respectively, we observe that

$$((\Delta T_1)^2)_{f_\alpha^e} = ((\Delta T_2)^2)_{f_\alpha^e} \tag{23}$$

$$((\Delta T_1)^2)_{f_\alpha^o} = ((\Delta T_2)^2)_{f_\alpha^o}$$

and that both satisfy their minimum uncertainty relation (21).

We define the two quadrature components of the backwardshift as

$$S_1 = \frac{1}{2}(T + T^*), \qquad S_2 = \frac{1}{2i}(T - T^*) \tag{24}$$

We observe the following facts:

$$(f_\alpha^e, T^*Tf_\alpha^e) = |\alpha|^2 \tanh_q|\alpha|^2 \tag{25}$$

$$(f_\alpha^o, T^*Tf_\alpha^o) = |\alpha|^2 \coth_q|\alpha|^2 \tag{26}$$

$$(f_\alpha^e, (T^2 + T^{*2})f_\alpha^e) = \alpha^2 + \bar{\alpha}^2 \tag{27}$$

$$(f_\alpha^o, (T^2 + T^{*2})f_\alpha^o) = \alpha^2 + \bar{\alpha}^2 \tag{28}$$

$$(f_\alpha^e, T^*f_\alpha^e) = 0 \tag{29}$$

$$(f_\alpha^o, T^*f_\alpha^o) = 0 \tag{30}$$

$$(f_\alpha^e, T^{*2}T^2f_\alpha^e) = |\alpha|^4 \tag{31}$$

$$(f_\alpha^o, T^{*2}T^2f_\alpha^o) = |\alpha|^4 \tag{32}$$

In calculating $(\Delta S_1)^2_{f_\alpha^e}$ we observe that

$$(f_\alpha^e, S_1^2 f_\alpha^e) = \frac{q^n}{4} + \frac{1}{2} r^2(\cos 2\phi + \tanh_q r^2) \tag{33}$$

$$(f_\alpha^e, S_1 f_\alpha^e) = 0 \tag{34}$$

where we have $\alpha = re^{i\theta}$.

Hence we have

$$(\Delta S_1)^2_{f_\alpha^e} = \langle S_1^2\rangle - \langle S_1\rangle^2 = \frac{q^n}{4} + \frac{1}{2} r^2(\cos 2\phi + \tanh_q r^2) \tag{35}$$

Similarly we have

$$(\Delta S_1)^2_{f_\alpha^o} = \langle S_2^2\rangle - \langle S_2\rangle^2 = \frac{q^n}{4} + \frac{1}{2} r^2(\cos 2\phi + \coth_q r^2) \tag{36}$$

Because $\tanh_q r^2 \simeq r^2$ and $\coth_q r^2 \simeq 1/r^2$, we see that $\tanh_q r^2 \leq 1$ and $\coth_q r^2 \geq 1$ if $r^2 \ll 1$. Thus a even coherent vector can exhibit squeezing, and an odd coherent vector nonsqueezing.

We calculate the correlation function $g^2(0)$ for even and odd coherent vector as follows:

$$g^2_{f_\alpha^e}(0) = \frac{(f_\alpha^e, T^{*2}T^2f_\alpha^e)}{(f_\alpha^e, T^*Tf_\alpha^e)^2} = \frac{1}{\tanh_q^2 r^2} \tag{37}$$

and

$$g^2_{f^o_\alpha}(0) = \frac{(f^o_\alpha, T^{*2}T^2 f^o_\alpha)}{(f^o_\alpha, T^*Tf^o_\alpha)^2} = \frac{1}{\coth^2_q r^2} \tag{38}$$

Because $\tanh_q r^2 \leq 1$ and $\coth_q r^2 \geq 1$, if $r^2 \ll 1$, we see that odd coherent vectors exhibit antibunching, but even coherent vectors cannot exhibit antibunching.

4. ORTHOGONAL-EVEN COHERENT VECTORS

Orthogonal-even coherent vectors are defined as a particular superposition of even coherent vectors

$$f^{oe}_\alpha = A[f^e_\alpha + f^e_{i\alpha}] \tag{39}$$

where f^e_α is given by (10).

To normalize we observe that

$$\begin{aligned} 1 = (f^{oe}_\alpha, f^{oe}_\alpha) &= A^2(f^e_\alpha + f^e_{i\alpha}, f^e_\alpha + f^e_{i\alpha}) \\ &= A^2[(f^e_\alpha, f^e_\alpha) + (f^e_\alpha, f^e_{i\alpha}) \\ &\quad + (f^e_{i\alpha}, f^e_\alpha) + (f^e_{i\alpha}, f^e_{i\alpha}) \end{aligned} \tag{40}$$

Again,

$$f^e_\alpha = B[f_\alpha + f_{-\alpha}]$$

with

$$B^2 = \frac{1}{2\,[1 + e_q(|\alpha|^2)^{-1} e_q(-|\alpha|^2)]} \tag{41}$$

We further observe that

$$(f^e_{i\alpha}, f^e_\alpha) = \frac{\cos_q|\alpha|^2}{\cosh_q|\alpha|^2} \tag{42}$$

From (40)–(42) we have

$$\begin{aligned} 1 &= A^2\left[1 + \frac{\cos_q|\alpha|^2}{\cosh_q|\alpha|^2} + \frac{\cos_q|\alpha|^2}{\cosh_q|\alpha|^2} + 1\right] \\ &= 2A^2\left[\frac{\cosh_q|\alpha|^2 + \cos_q|\alpha|^2}{\cosh_q|\alpha|^2}\right] \end{aligned} \tag{43}$$

Hence the normalization constant A is given by

$$A^2 = \frac{\cosh_q|\alpha|^2}{2[\cosh_q|\alpha|^2 + \cos_q|\alpha|^2]} \tag{44}$$

4.1. Quadrature Variance for Orthogonal-Even Coherent Vectors

We take two quadrature components S_1 and S_2 given by (24).
We know that

$$(\Delta S_1)^2 = \langle S_1^2 \rangle - \langle S_1 \rangle^2 \tag{45}$$

To calculate (45) for orthogonal-even coherent vectors we observe the following facts:

$$\begin{aligned}
(f_\alpha^e, f_{i\alpha}^o) &= 0 \\
(f_{i\alpha}^e, f_\alpha^o) &= 0 \\
(f_{i\alpha}^e, f_\alpha^e) &= \frac{\cos_q|\alpha|^2}{\cosh_q|\alpha|^2} \\
(f_\alpha^o, f_{i\alpha}^o) &= i\frac{\sin_q|\alpha|^2}{\sinh_q|\alpha|^2} \\
(f_\alpha^e, Tf_{i\alpha}^e) &= 0 \\
(f_{i\alpha}^e, Tf_\alpha^e) &= 0 \\
(f_\alpha^{oe}, Tf_\alpha^{oe}) &= 0 \\
(f_\alpha^{oe}, T^2 f_\alpha^{oe}) &= 0
\end{aligned} \tag{46}$$

Hence we have

$$(\Delta S_1)^2_{f_\alpha^{oe}} = \langle S_1^2 \rangle - \langle S_1 \rangle^2 = \frac{1}{2} r^2 \frac{\sinh_q r^2 - \sin_q r^2}{\cosh_q r^2 + \cos_q r^2} + \frac{q^n}{4} \tag{47}$$

where we have $\alpha = re^{i\theta}$.

As the right-hand side of (47) is ≤ 1 if $r^2 \ll 1$, we see that orthogonal-even coherent vectors can exhibit squeezing.

We calculate correlation function $g^2(0)$ for an orthogonal-even coherent vector as follows:

$$g^2_{f_\alpha^{oe}}(0) = \frac{(f_\alpha^{oe}, T^{*2} T^2 f_\alpha^{oe})}{(f_\alpha^{oe}, T^* T f_\alpha^{oe})^2} = \frac{[\cosh_q r^2 - \cos_q r^2][\cosh_q r^2 + \cos_q r^2]}{[\sinh_q r^2 - \sin_q r^2]^2} \tag{48}$$

We observe that $g^2_{f_\alpha^{oe}}(0) \geq 1$ if $r^2 \ll 1$.

Thus an orthogonal-even coherent vector cannot exhibit antibunching.

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